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STRONG ELLIPTICITY OF COMPARISON SOLIDS IN ELASTOPLASTICITY WITH VOLUMETRIC NON-ASSOCIATIVITY

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Abstract—Reference is made to non-associative, time independent infinitesimal elastoplasticity with non-associative flow rule. The non-associativity is restricted to the volumetric component of the plastic flow. Loss of strong ellipticity is shown to occur before the snap-back modulus is reached. An analytical solution for the condition of loss of strong ellipticity is obtained for the best chosen comparison solid of the family introduced by Raniecki (1979, *Bull. Acad. Polon. Sci.* **XXVII**, 391–399). Finally, this solution is shown to coincide with the loss of strong ellipticity in the comparison solid "in loading". Therefore, for volumetric non-associative flow-rules, strong ellipticity is lost simultaneously in the best chosen comparison solid of Raniecki and in the comparison solid "in loading".

1. INTRODUCTION

The condition of strong ellipticity (S-E in the following) represents a local stability criterion that excludes strain localization (Hill and Hutchinson, 1975; Rice, 1976; Rudnicki and Rice, 1975; Vardoulakis, 1976) but may still be verified even when the second order work is not positive definite (Maier and Hueckel, 1979; Villaggio, 1968). The S-E condition requires the positive definiteness of the acoustic tensor corresponding to the constitutive operator. This condition, well known in the context of finite elasticity (Truesdell and Noll, 1965), has been investigated in the context of plasticity mainly for the case of the associative flow law [see, e.g. Hill (1962) and Thomas (1961)] and only recently for non-associative flow-rules (Ryzhak, 1987).

This paper is addressed to the infinitesimal theory of (inviscid) elastoplasticity in the presence of deviatoric normality, i.e. only the volumetric component of the plastic flow is assumed to be non-associative. This context has been widely explored from the point of view of local and integral stability criteria (Bigoni and Hueckel, 1990; Loret *et al.*, 1990; Mróz, 1963; Needleman, 1979; Nemat-Nasser and Shokooh, 1980; Rudnicki and Rice, 1975; Rudnicki, 1977). In this paper it is shown that S-E is necessarily lost in the comparison solid "in loading" before the snap-back plastic modulus is reached. Moreover, an analytical solution is found for the loss of S-E in the best chosen comparison solid of the family introduced by Raniecki (1979) [see also Raniecki and Bruhns (1981)]. By using this solution and bearing on the hypothesis of deviatoric normality, it is finally shown that the loss of S-E occurs at the same value of the (critical) plastic modulus for the comparison solid "in loading" and for the best chosen comparison solid of Raniecki's (1979) family. A simple application to the model of Rudnicki and Rice (1975) closes the paper. The case of generic non-associative elastoplasticity is dealt with elsewhere (Bigoni and Zaccaria, 1992).

2. PROBLEM FORMULATION

The notation of modern continuous mechanics is used (Gurtin, 1981). A tensor **A** is a linear transformation over a three-dimensional inner product space $\mathscr{V}$. Lin denotes the set of all tensors and Sym the subset of symmetric tensors. A fourth-order tensor $\mathbb{A}$ is a linear transformation over Lin (or Sym). The symbol $\otimes$ represents the tensor product over an inner product space (including Lin). The symbol $\langle \cdot \rangle$ denotes the McAuley brackets, i.e. the operator $\mathbb{R} \rightarrow \mathbb{R}^+ \cup \{0\}$, $\alpha \rightarrow \langle \alpha \rangle = \sup\{\alpha, 0\}$.

Reference is made to the incremental elastoplastic constitutive law with two tensorial zones, relating the rate of Cauchy stress $\dot{\mathbf{T}}$ to the velocity of deformation $\mathbf{D}$:

$$\dot{\mathbf{T}} = \mathbb{E}[\mathbf{D}] - \frac{\langle \mathbf{D}\cdot\mathbb{E}[\mathbf{Q}]\rangle}{g}(\mathbb{E}[\mathbf{Q}]+3\xi\kappa\mathbf{I}), \tag{1}$$

where $\mathbf{Q}$ is the yield surface gradient, g the plastic modulus, κ the elastic bulk modulus, ξ specifies the degree of non-associativity and $\mathbb{E}$ is the elastic tensor, assumed isotropic and positive definite:

$$\mathbb{E} = \lambda\mathbf{I}\otimes\mathbf{I}+2\mu\mathbb{I}, \tag{2}$$

in which $\mathbb{I}$ is the fourth-order identity tensor and λ and μ are the Lamé moduli related to the bulk modulus by $3\kappa = 3\lambda+2\mu$.

It is worth noting, from the constitutive equation (1), that the case of associative plasticity is recovered for $\xi = 0$. Moreover, the plastic modulus g is related to the hardening modulus h by:

$$g = h+\mathbf{Q}\cdot\mathbb{E}[\mathbf{Q}]+3\xi\kappa\,\mathrm{tr}\,\mathbf{Q}. \tag{3}$$

When h is positive, the hardening regime is described, whereas negative values of h model the softening behavior. When g is zero the "snap-back" modulus is reached and negative values of g describe the sub-critical behavior [in the sense defined by Maier and Hueckel (1979) and Borrè and Maier (1989)].

On the basis of the constitutive equation (1), the incrementally linear comparison solid "in loading" is defined through the introduction of the fourth order constitutive tensor $\mathbb{N}$:

$$\mathbb{N} = \mathbb{E}-\frac{(\mathbb{E}[\mathbf{Q}]+3\xi\kappa\mathbf{I})\otimes\mathbb{E}[\mathbf{Q}]}{g}. \tag{4}$$

The family of comparison solids introduced by Raniecki (1979) depends continuously on a strictly positive scalar parameter γ and is specified by the fourth-order constitutive tensor $\mathbb{M}$ as follows:

$$\mathbb{M}(\gamma) = \mathbb{E}-\frac{\mathbb{E}[(1+\gamma)\mathbf{Q}+\xi\mathbf{I}]\otimes\mathbb{E}[(1+\gamma)\mathbf{Q}+\xi\mathbf{I}]}{4\gamma g}. \tag{5}$$

The acoustic tensors $\mathbf{A}_N(\mathbf{n})$, $\mathbf{A}_M(\mathbf{n})$ and $\mathbf{A}_E(\mathbf{n})$ are now introduced corresponding to the solid (4), to the family of solids (5) and to the elastic solid, respectively:

$$\mathbf{A}_N(\mathbf{n})\mathbf{m} = \mathbb{N}[\mathbf{m}\otimes\mathbf{n}]\mathbf{n}, \qquad \mathbf{A}_M(\gamma,\mathbf{n})\mathbf{m} = \mathbb{M}(\gamma)[\mathbf{m}\otimes\mathbf{n}]\mathbf{n}, \qquad \mathbf{A}_E(\mathbf{n})\mathbf{m} = \mathbb{E}[\mathbf{m}\otimes\mathbf{n}]\mathbf{n}. \tag{6–8}$$

Throughout the paper, when misunderstandings are excluded, the dependence on $\mathbf{n}$ of the acoustic tensors (6)–(8) will be omitted.

For the comparison solid "in loading" (4) the condition of strong ellipticity is written as:

$$\mathbf{m}\cdot\mathbf{A}_N(\mathbf{n})\mathbf{m} > 0, \quad \forall\ \mathbf{n},\mathbf{m}\in\mathcal{V}. \tag{9}$$

Analogously, for the generic solid of the family (5) introduced by Raniecki, the condition of strong ellipticity reads:

$$\mathbf{m}\cdot\mathbf{A}_M(\gamma,\mathbf{n})\mathbf{m} > 0, \quad \forall\ \mathbf{n},\mathbf{m}\in\mathcal{V}. \tag{10}$$

Note that the comparison theorem of Raniecki (1979) implies:

$$\mathbf{m}\cdot\mathbf{A}_N(\mathbf{n})\mathbf{m} \geq \mathbf{m}\cdot\mathbf{A}_M(\gamma,\mathbf{n})\mathbf{m}, \quad \forall\, \mathbf{n},\mathbf{m}\in\mathscr{V}. \tag{11}$$

Finally, it can be observed that, due to the positive definiteness of $\mathbb{E}$, the elastic acoustic tensor is always positive definite.

3. CRITICAL PLASTIC MODULI FOR THE LOSS OF S-E

The condition of loss of strong ellipticity is expressed in this section in terms of a critical value of the plastic modulus for the comparison solids "in loading" and for Raniecki's family of comparison solids. It is also shown that the loss of S-E always occurs before the snap-back value of the plastic modulus is reached.

3.1. *Comparison solid "in loading"*

The condition of loss of S-E may be obtained, for a given $\mathbf{n}$, by equating to zero the solution of the following constrained minimization problem:

$$\min_{\mathbf{m}\in\mathscr{V}} \{\mathbf{m}\cdot\mathbf{A}_E(\mathbf{n})\mathbf{m}-\mathbf{m}\cdot\mathbb{E}[\mathbf{Q}]\mathbf{n}-3\xi\kappa\mathbf{m}\cdot\mathbf{n}\}, \tag{12}$$

subject to:

$$\mathbf{m}\cdot\mathbb{E}[\mathbf{Q}]\mathbf{n} = g. \tag{13}$$

By introducing the function

$$\mathscr{L}(\mathbf{m},g,\chi) = \mathbf{m}\cdot\mathbf{A}_E\mathbf{m}-(1-\chi)\mathbf{m}\cdot\mathbb{E}[\mathbf{Q}]\mathbf{n}-3\xi\kappa\mathbf{m}\cdot\mathbf{n}-\chi g, \tag{14}$$

where χ is a Lagrangian multiplier, the problem (12)–(13) becomes

$$\min_{\mathbf{m}\in\mathscr{V},\chi\in\mathbb{R}} \mathscr{L}(\mathbf{m},\chi,g^N) = 0, \tag{15}$$

in which g^N denotes the critical value of the plastic modulus for the loss of S-E, at a given value of $\mathbf{n}$. The solution of (15) (see Appendix A) is given by:

$$\mathbf{m}^N(\mathbf{n}) = \tfrac{1}{2}[(1+\chi^N)\mathbf{m}_a+\bar{v}\xi\mathbf{n}], \qquad g^N(\mathbf{n}) = \tfrac{1}{2}[(1+\chi^N)\mathbf{m}_a\cdot\mathbf{A}_E\mathbf{m}_a+\bar{v}\xi\mathbf{n}\cdot\mathbf{A}_E\mathbf{m}_a], \tag{16, 17}$$

$$\chi^N(\mathbf{n}) = \sqrt{(\mathbf{m}_a+\bar{v}\xi\mathbf{n})\cdot\mathbf{A}_E(\mathbf{m}_a+\bar{v}\xi\mathbf{n})/\mathbf{m}_a\cdot\mathbf{A}_E\mathbf{m}_a}, \tag{18}$$

where:

$$\mathbf{m}_a = \mathbf{A}_E^{-1}\mathbb{E}[\mathbf{Q}]\mathbf{n}, \qquad \bar{v} = \frac{1+v}{1-v}. \tag{19, 20}$$

A substitution of (18) into (17) yields:

$$g^N(\mathbf{n}) = \tfrac{1}{2}[(\mathbf{m}_a+\bar{v}\xi\mathbf{n})\cdot\mathbf{A}_E\mathbf{m}_a+\sqrt{(\mathbf{m}_a+\bar{v}\xi\mathbf{n})\cdot\mathbf{A}_E(\mathbf{m}_a+\bar{v}\xi\mathbf{n})\mathbf{m}_a\cdot\mathbf{A}_E\mathbf{m}_a}]. \tag{21}$$

Therefore, the Cauchy–Buniakovskiĭ–Schwarz inequality implies that

$$g^N(\mathbf{n}) \geq 0 \tag{22}$$

and

$$g^N(\mathbf{n}) = 0 \Leftrightarrow \exists\,\alpha\in\mathbb{R}^+ : \mathbf{m}_a+\bar{v}\xi\mathbf{n} = -\alpha\mathbf{m}_a. \tag{23}$$

In a continuous loading path, S-E is lost when the plastic modulus becomes equal to the critical one, that corresponds to the maximum of $g^N(\mathbf{n})$ over all directions $\mathbf{n}$. Therefore, using the expression (21), the critical value of the plastic modulus $\mathscr{g}_{SE}^{N}$ for the loss of S-E is obtained as the solution to the constrained maximization problem:

$$\mathscr{g}_{SE}^{N} = \max_{\mathbf{n}\in\mathscr{V}} \mathscr{g}^{N}(\mathbf{n}), \tag{24}$$

subject to

$$\mathbf{n}\cdot\mathbf{n} = 1. \tag{25}$$

The direction for which $\mathscr{g}_{SE}^{N}$ is maximized will be denoted in the following as $\mathbf{n}_{SE}^{N}$.

If (19) is substituted into (23), the condition for which (23) is satisfied for every $\mathbf{n}$ yields:

$$\mathscr{g}_{SE}^{N} = 0, \quad \Leftrightarrow \exists\,\alpha\in(0,1): \mathbf{Q} = -\alpha\xi\mathbf{I}. \tag{26}$$

Therefore, loss of S-E may correspond with the snap-back modulus $\mathscr{g} = 0$ if and only if tensor $\mathbf{Q}$ is spherical and the plastic flow has the same direction in the stress space as the yield surface gradient, but is directed inside the yield function. This kind of flow rule is usually (Runesson and Mróz, 1989) not accepted and therefore loss of S-E is ensured before the snap-back modulus is reached.

3.2. *Comparison solids of the family introduced by Raniecki*

The condition of loss of S-E may, for a given $\mathbf{n}$, be obtained by using a procedure quite similar to that of the previous case. The following result is obtained:

$$\mathbf{m}^{M}(\gamma,\mathbf{n}) = (1+\gamma)\mathbf{m}_{a} + \bar{v}\xi\mathbf{n}, \tag{27}$$

$$\mathscr{g}^{M}(\gamma,\mathbf{n}) = \frac{1}{2}\left[(\mathbf{m}_{a}+\xi\bar{v}\mathbf{n})\cdot\mathbf{A}_{E}\mathbf{m}_{a} + \frac{\gamma}{2}\mathbf{m}_{a}\cdot\mathbf{A}_{E}\mathbf{m}_{a} + \frac{1}{2\gamma}(\mathbf{m}_{a}+\xi\bar{v}\mathbf{n})\cdot\mathbf{A}_{E}(\mathbf{m}_{a}+\xi\bar{v}\mathbf{n})\right], \tag{28}$$

and the Lagrangean multiplier proves to be, in this case, equal to 1.

Using the expression (28), the critical value of the plastic modulus $\mathscr{g}^{M}(\gamma)$ for the loss of S-E in the generic solid (5) is obtained as the solution of the constrained maximization problem:

$$\mathscr{g}^{M}(\gamma) = \max_{\mathbf{n}\in\mathscr{V}} \mathscr{g}^{M}(\gamma,\mathbf{n}), \tag{29}$$

subject to

$$\mathbf{n}\cdot\mathbf{n} = 1. \tag{30}$$

The S-E condition is lost in all solids of the family introduced by Raniecki at the minimum value $\mathscr{g}_{SE}^{M}$ of $\mathscr{g}^{M}(\gamma)$:

$$\mathscr{g}_{SE}^{M} = \inf_{\gamma\in\mathbb{R}^{+}} \mathscr{g}^{M}(\gamma). \tag{31}$$

The value γ_{SE} of γ that satisfies (31) defines the "best chosen comparison solid" of Raniecki. The direction for which $\mathscr{g}_{x}^{M}(\gamma,\mathbf{n})$ is first maximized in respect to $\mathbf{n}$ and then minimized in respect to γ, will be denoted in the following as $\mathbf{n}_{SE}^{M}$.

Now, if $\mathscr{g}^{M}(\gamma,\mathbf{n})$ is minimized with respect to γ, at a constant value of $\mathbf{n}$, it is readily obtained that $\mathscr{g}^{M}(\gamma,\mathbf{n})$ possesses a minimum in correspondence of $\gamma = \chi^{N}(\mathbf{n})$ and:

$$\mathscr{g}^{M}(\chi^{N}(\mathbf{n}),\mathbf{n}) = \mathscr{g}^{N}(\mathbf{n}). \tag{32}$$

Thus the problem (29)–(31) can be written in the form:

$$g_{SE}^{M} = \inf_{\gamma \in R^+} \max_{n \in \mathscr{V}} g^{M}(\gamma, \mathbf{n}), \quad \text{subject to } \mathbf{n}\cdot\mathbf{n} = 1, \tag{33}$$

whereas the problem (24)–(25) can be written in the dual form:

$$g_{SE}^{N} = \max_{n \in \mathscr{V}} \min_{\gamma \in R^+} g^{M}(\gamma, \mathbf{n}), \quad \text{subject to } \mathbf{n}\cdot\mathbf{n} = 1. \tag{34}$$

4. ANALYTICAL SOLUTION FOR THE CRITICAL PLASTIC MODULUS FOR LOSS OF S-E IN THE COMPARISON SOLIDS OF RANIECKI

In this section an analytical solution for the loss of S-E in a generic solid of Raniecki is derived. The solution is expressed in terms of a critical value $g^M(\gamma)$ of the plastic modulus and a critical direction $\mathbf{n}^M(\gamma)$.

By introducing the Lagrangean multiplier η, the constrained maximization problem (29)–(30) is reduced to:

$$\max_{n \in \mathscr{V}} \{ g^{M}(\gamma, \mathbf{n}) + \eta(\mathbf{n}\cdot\mathbf{n} - 1)\}, \tag{35}$$

where $g^M(\gamma, \mathbf{n})$ can be written, substituting (2) into (8), as:

$$g^{M}(\gamma, \mathbf{n}) = \frac{G}{2\gamma}\left\{2\mathbf{n}\cdot\mathbf{P}^2\mathbf{n} - \frac{1}{1-v}(\mathbf{n}\cdot\mathbf{P}\mathbf{n})^2 + \frac{2v}{1-v}(\mathbf{n}\cdot\mathbf{P}\mathbf{n})\,\mathrm{tr}\,\mathbf{P} + \frac{v^2}{(1-v)(1-2v)}\mathrm{tr}^2\,\mathbf{P}\right\}, \tag{36}$$

in which

$$\mathbf{P} = (1+\gamma)\mathbf{Q} + \xi\mathbf{I}. \tag{37}$$

The problem (35) can be solved (see Appendix B) in a similar manner as in the case of strain localization solved by Bigoni and Hueckel (1991). The solution of problem (35) is given by one of the following directions of $\mathbf{n}$ (expressed in components in the principal reference system of $\mathbf{Q}$):

$$n_k = 0, \quad n_l^2 = \langle 1 - \langle -\alpha_m \rangle\rangle, \quad n_m^2 = \langle 1 - \langle -\alpha_l \rangle\rangle, \tag{38}$$

for every permutation (k, l, m) of $(1, 2, 3)$, where:

$$\alpha_l = \frac{Q_l + vQ_k}{Q_m - Q_l} + \frac{(1+v)\xi}{(1+\gamma)(Q_m - Q_l)}, \qquad \alpha_m = \frac{Q_m + vQ_k}{Q_l - Q_m} + \frac{(1+v)\xi}{(1+\gamma)(Q_l - Q_m)}, \tag{39, 40}$$

in which Q_k, Q_l and Q_m denote the eigenvalues of $\mathbf{Q}$. Note that if $Q_l = Q_m$, the expressions (35) along with the following still remain valid as limits $\alpha_l \to +\infty$ and $\alpha_m \to -\infty$ or $\alpha_l \to -\infty$ and $\alpha_m \to +\infty$. The problem (35) is reduced to:

$$g^{M}(\gamma) = \max_{k=1,2,3} g^{M}(\gamma, k), \tag{41}$$

where:

$$g^{M}(\gamma, k) = \frac{G}{2}\left[\gamma\mathscr{A}(\gamma, k) + \frac{1}{\gamma}\mathscr{B}(\gamma, k) + 2\mathscr{C}(\gamma, k)\right] \tag{42}$$

and

$$\mathscr{A}(\gamma,k) = \mathbf{Q}\cdot\mathbf{Q}+\frac{v}{1-2v}\operatorname{tr}^2\mathbf{Q}-(1+v)Q_k^2 - \frac{1}{1-v}\left[\frac{\langle\alpha_m\rangle}{\alpha_m}(Q_m+vQ_k)^2+\frac{\langle\alpha_l\rangle}{\alpha_l}(Q_l+vQ_k)^2\right], \quad (43)$$

$$\mathscr{B}(\gamma,k) = \mathscr{A}(\gamma,k)+\xi\left\{\frac{1+v}{1-2v}(2\operatorname{tr}\mathbf{Q}+3\xi)-(1+v)(2Q_k+\xi) - \bar{v}\left[\frac{\langle\alpha_m\rangle}{\alpha_m}(2(Q_m+vQ_k)+(1+v)\xi)+\frac{\langle\alpha_l\rangle}{\alpha_l}(2(Q_l+vQ_k)+(1+v)\xi)\right]\right\}, \quad (44)$$

$$\mathscr{C}(\gamma,k) = \mathscr{A}(\gamma,k)+\xi\left\{\frac{1+v}{1-2v}\operatorname{tr}\mathbf{Q}-(1+v)Q_k - \bar{v}\left[\frac{\langle\alpha_m\rangle}{\alpha_m}(Q_m+vQ_k)+\frac{\langle\alpha_l\rangle}{\alpha_l}(Q_l+vQ_k)\right]\right\}, \quad (45)$$

in which if one of the $\alpha_i (i = l, m)$ parameters is zero, $\langle\alpha_i\rangle/\alpha_i$ is to be replaced by 0 or 1 indifferently. This indifference implies that the functions $\mathbb{R}^+ \to \mathbb{R}$, $\gamma \to g^M(\gamma,k)$, $k = 1,2,3$, are continuous and therefore the function (41) also results to be continuous.

It is worth noting that the functions $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ are unchanged by the choice of l and m. Moreover, by comparing (42) with (28), it is seen that $\mathscr{A}$ and $\mathscr{B}$ are given, in correspondence of (38), by:

$$\mathscr{A}(\gamma,k) = \frac{1}{G}\mathbf{m}_a(k)\cdot\mathbf{A}_E\mathbf{m}_a(k), \quad (46)$$

$$\mathscr{B}(\gamma,k) = \frac{1}{G}[\mathbf{m}_a(k)+\xi\bar{v}\mathbf{n}(k)]\cdot\mathbf{A}_E[\mathbf{m}_a(k)+\xi\bar{v}\mathbf{n}(k)], \quad (47)$$

where $\mathbf{n}(k)$ is the solution (38) and $\mathbf{m}_a(k) = \mathbf{m}_a(\mathbf{n}(k))$.

From (46)–(47) it is concluded that:

$$\mathscr{A}(\gamma,k) \geq 0 \quad \text{and} \quad \mathscr{B}(\gamma,k) \geq 0. \quad (48)$$

Moreover, from (43)–(45) it is seen that the functions $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ are independent of γ inside the following intervals of $\mathbb{R}^+$:

$$\mathscr{I}_{[0,1]}(k) = \{\gamma\in\mathbb{R}^+/\langle\alpha_l\rangle = \langle\alpha_m\rangle = 0\}, \qquad \mathscr{I}^l_{(-\infty,0]}(k) = \{\gamma\in\mathbb{R}^+/\langle\alpha_l\rangle \geq 0\}, \quad (49, 50)$$

$$\mathscr{I}^m_{(-\infty,0]}(k) = \{\gamma\in\mathbb{R}^+/\langle\alpha_m\rangle \geq 0\}. \quad (51)$$

From (49)–(51), note that the dependence on k is through α_l and α_m and therefore the following implications hold true:

$$\gamma\in\mathscr{I}_{[0,1]}(k) \Rightarrow -\alpha_l, -\alpha_m\in[0,1], \quad \gamma\in\mathscr{I}^l_{(-\infty,0]}(k) \Rightarrow -\alpha_l\in(-\infty,0], -\alpha_m\in[1,+\infty), \quad (52, 53)$$

$$\gamma\in\mathscr{I}^m_{(-\infty,0]}(k) \Rightarrow -\alpha_l\in[1,+\infty), -\alpha_m\in(-\infty,0]. \quad (54)$$

In the case where tensor $\mathbf{Q}$ has one multiple eigenvalue, all the above formulae still hold although they may be written in a simpler way. Moreover, if, for example, $Q_l = Q_m$(41) reduces to:

$$\mathcal{g}^{M}(\gamma) = \mathcal{g}^{M}(\gamma, l) = \mathcal{g}^{M}(\gamma, m), \tag{55}$$

and the maximization is not requested.

5. COINCIDENCE OF CRITICAL PLASTIC MODULI FOR LOSS OF S-E IN THE COMPARISON SOLIDS

The coincidence between the critical plastic moduli for the loss of S-E in the comparison solid "in loading" and in the best chosen comparison solid of Raniecki will be proved in the following. The result, which follows from solving problem of Section 4, is obtained with a proofing technique that, due to the hypothesis of deviatoric normality, is not based on the restrictive conditions of convex analysis [as in Bigoni and Zaccaria (1992)].

5.1. *Explicit solution for loss of S-E*

The function $\mathcal{g}^{M}(\gamma)$ defined by (41) is formed by pieces of functions (42) in which the coefficients $\mathcal{A} \geqslant 0$, $\mathcal{B} \geqslant 0$ and $\mathcal{C}$ are constant, i.e. the function is the finite union of pieces of functions in the form:

$$\mathbb{R}^{+} \to \mathbb{R}, \quad \gamma \to \frac{G}{2}\left[\gamma\bar{\mathcal{A}} + \frac{1}{\gamma}\bar{\mathcal{B}} + 2\bar{\mathcal{C}}\right], \quad \bar{\mathcal{A}}, \bar{\mathcal{B}} \geqslant 0. \tag{56}$$

If $\bar{\mathcal{A}}, \bar{\mathcal{B}} \neq 0$, the function (56) reaches a minimum in the point

$$\bar{\gamma} = \sqrt{\bar{\mathcal{B}}/\bar{\mathcal{A}}}. \tag{57}$$

Moreover, the function $\mathcal{g}^{M}(\gamma)$ will be shown to be coercive:

$$\lim_{\gamma \to +\infty} \mathcal{g}^{M}(\gamma) = \lim_{\gamma \to 0} \mathcal{g}^{M}(\gamma) = +\infty \tag{58}$$

and to have a continuous first derivative. Thus $\mathcal{g}^{M}(\gamma)$ has a minimum in correspondence to the value γ_{SE} of γ in the form (57), where

$$\bar{\mathcal{A}} = \mathcal{A}(\gamma_{SE}, k), \quad \bar{\mathcal{B}} = \mathcal{B}(\gamma_{SE}, k), \tag{59}$$

with k satisfying

$$\mathcal{g}^{M}(\gamma_{SE}, k) \geqslant \mathcal{g}^{M}(\gamma_{SE}, i), \quad i = 1, 2, 3. \tag{60}$$

Therefore, the determination of the best chosen comparison solid is simply reduced to finding among all (maximum 9) values of γ in the form (57), the value γ_{SE} that satisfies (59)–(60).

In order to prove the coercivity of the function $\mathcal{g}^{M}(\gamma)$, let us note that this function necessarily has the form (56) in a neighborhood of $+\infty$ and 0, therefore:

$$\lim_{\gamma \to +\infty} \mathcal{g}^{M}(\gamma) \neq +\infty \Rightarrow \bar{\mathcal{A}} = 0, \tag{61}$$

and, by the definition (41) of $\mathcal{g}^{M}(\gamma)$:

$$\bar{\mathcal{A}} = 0 \Rightarrow [\mathcal{A}(\gamma, i) = 0, \quad i = 1, 2, 3]. \tag{62}$$

If (58) does not hold, (61) and (62) imply the condition $\mathcal{A}(\gamma, i) = 0$, $i = 1, 2, 3$, that, using the definition (43) of $\mathcal{A}$, leads to a contradiction. An analogous scheme can be used to prove the coercivity in respect of $\gamma \to 0$.

In the following, the continuity of the first derivative of the functions $g^{M}(\gamma, k)$ and $g^{M}(\gamma)$ is shown in two Lemmas.

Lemma 1. *The functions* $\mathbb{R}^+ \to \mathbb{R}$, $\gamma \to g^{M}(\gamma, k)$, $k = 1, 2, 3$ *have continuous first derivatives.*

Proof. From the definition (42), the continuity of the function $\mathbb{R}^+ \to \mathscr{R}$, $\gamma \to g^{M}(\gamma, k)$ is certain for values of γ internal to the intervals (49)–(51). Now, the frontier points of the intervals (49)–(51) are to be checked. Only the intersections between the intervals (49) and (50) and (49) and (51) are to be considered, in fact:

$$\mathscr{I}^{\mathrm{l}}_{(-\infty,0]} \cap \mathscr{I}^{\mathrm{m}}_{(-\infty,0]} = \varnothing, \tag{63}$$

unless $Q_{\mathrm{l}} = Q_{\mathrm{m}}$, which implies $\mathscr{I}^{\mathrm{l}}_{(-\infty,0]} = \mathscr{I}^{\mathrm{m}}_{(-\infty,0]} = \mathbb{R}^+$. Let us suppose

$$\bar{\gamma} \in \mathscr{I}^{\mathrm{l}}_{(-\infty,0]} \cap \mathscr{I}_{[0,1]}, \tag{64}$$

therefore, from (52)–(53) it is concluded that $\alpha_{\mathrm{l}} = 0$ and from (39) that:

$$\bar{\gamma} = -1 - \frac{\xi(1+v)}{Q_{\mathrm{l}} + vQ_{\mathrm{k}}}. \tag{65}$$

By deriving the function (42) with respect to γ on the intervals $\mathscr{I}^{\mathrm{l}}_{(-\infty,0]}$ and $\mathscr{I}_{[0,1]}$ and by substituting (65) into the obtained expressions, continuity is readily verified. The case of $\bar{\gamma} \in \mathscr{I}^{\mathrm{m}}_{(-\infty,0]} \cap \mathscr{I}_{[0,1]}$ is quite analogous.

Lemma 2. *The function* $\mathbb{R}^+ \to \mathbb{R}$, $\gamma \to g^{M}(\gamma)$, *defined by* (41), *has a continuous first derivative.*

Proof. In the case where the tensor $\mathbf{Q}$ has one multiple eigenvalue, Lemma 1, along with (55), ensures the validitiy of Lemma 2. In the following, only the cases of distinct eigenvalues are considered. The function (41) is, by definition, the union of pieces of the three functions $g^{M}(\gamma, i)$ ($i = 1, 2, 3$). These functions are continuous and, from Lemma 1, have continuous first derivatives. Therefore a possible point of discontinuity coincides with an intersection point $\bar{\gamma}$ of two functions $\gamma \to g^{M}(\gamma, p)$ and $\gamma \to g^{M}(\gamma, q)$ (with $p \neq q$), where $g^{M}(\bar{\gamma}, p) = g^{M}(\bar{\gamma}, q)$. Now, all the possible points of intersection are examined.

In Appendix C, it is shown that the points of the interval

$$\mathscr{I}^{pq}_{(-\infty,0]} = \mathscr{I}^{p}_{(-\infty,0]}(q) \cap \mathscr{I}^{q}_{(-\infty,0]}(p), \tag{66}$$

are points of intersection between $g^{M}(\gamma, p)$ and $g^{M}(\gamma, q)$ and this interval is empty or contains infinite points. Therefore, the first derivative of functions $g^{M}(\gamma, p)$ and $g^{M}(\gamma, q)$ coincide in the interval $\mathscr{I}^{pq}_{(-\infty,0]}$, including its frontier points.

In Appendix D it is shown that intersection points not belonging to the interval (66) satisfy:

$$g^{M}(\bar{\gamma}, t) > g^{M}(\bar{\gamma}, p) = g^{M}(\bar{\gamma}, q), \tag{67}$$

where (t, p, q) is a permutation of $(1, 2, 3)$. Taking into account the definition (41) of $g^{M}(\gamma)$, (67) implies that the intersection points do not belong to the function $g^{M}(\gamma)$.

5.2. *Concidence of the critical plastic moduli for the comparison solids*

The critical plastic modulus for the loss of S-E in the best chosen comparison solid of Raniecki coincides with the critical plastic modulus for the loss of S-E in the comparison solid "in loading". In order to prove this result, let us note that a substitution of (46)–(47) into (57) and a comparison with definition (18) yields:

$$\chi^{N}(\mathbf{n}_{SE}^{M}) = \gamma_{SE}. \tag{68}$$

By definition (24) of $\mathscr{g}_{SE}^{N}$, the following inequality holds true

$$\mathscr{g}_{SE}^{N} = \mathscr{g}^{N}(\mathbf{n}_{SE}^{N}) \geq \mathscr{g}^{N}(\mathbf{n}_{SE}^{M}), \tag{69}$$

but (32) implies $\mathscr{g}^{N}(\mathbf{n}_{SE}^{M}) = \mathscr{g}^{M}(\chi^{N}(\mathbf{n}_{SE}^{M}), \mathbf{n}_{SE}^{M})$ and through (68), (69) becomes

$$\mathscr{g}_{SE}^{N} \geq \mathscr{g}^{M}(\gamma_{SE}, \mathbf{n}_{SE}^{M}) = \mathscr{g}_{SE}^{M}. \tag{70}$$

The comparison theorem of Raniecki implies (11) and thus $\mathscr{g}_{SE}^{M} \geq \mathscr{g}_{SE}^{N}$. Therefore, from (70) it is concluded that $\mathscr{g}_{SE}^{M} = \mathscr{g}_{SE}^{N}$.

6. EXAMPLE: THE RUDNICKI AND RICE MODEL

An application is made to the model proposed by Rudnicki and Rice (1975), in which the Drucker–Prager yield function is used in conjunction with a non-associative flow rule, allowing for deviatoric normality. For this model, the yield surface gradient results in the form:

$$\mathbf{Q} = \frac{\operatorname{dev}\mathbf{T}}{(2J_2)^{1/2}} + \frac{\alpha}{3}\mathbf{I}, \tag{71}$$

where α is a non-negative constitutive parameter, dev $\mathbf{T}$ is the deviatoric stress and J_2 is its second invariant. With the procedure shown in Section 5, the critical hardening modulus h_{SE} for the loss of S-E has been evaluated for all the stress points lying on the generic deviatoric section of the yield function. In Fig. 1, the stress point is identified through

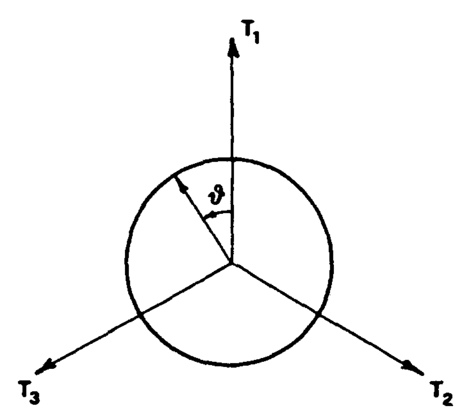

Fig. 1. Deviatoric section of the yield function with Lode's angle.

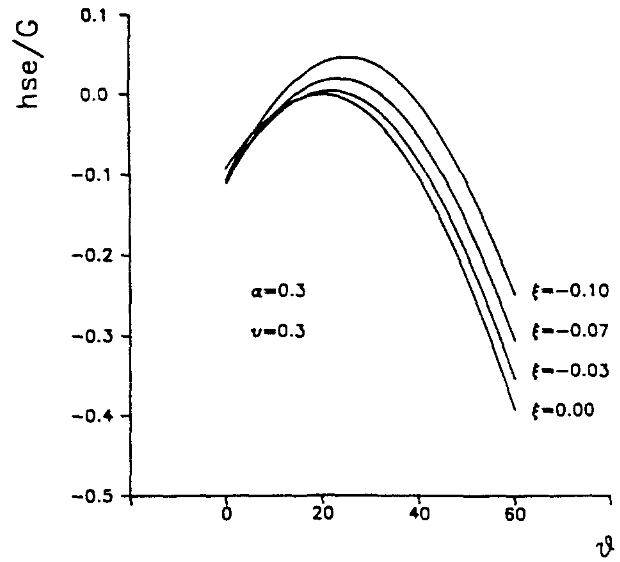

Fig. 2. Critical hardening modulus (normalized in respect to the elastic shear modulus) vs Lode's angle (in degrees).

Lode's angle ϑ. Figures 2, 3 and 4, refer to the cases of $\alpha = 0.3$, 0.6, 0.9 and $\nu = 0.3$. The range of the parameter ξ is -0.3–0. In the case $\xi = 0$, the associative flow rule is recovered and therefore the loss of S-E coincides with the loss of ellipticity and the same results as in Rudnicki and Rice (1975) are found. In particular, in the case of the associative flow rule, the critical hardening modulus is never positive (i.e. loss of S-E is excluded in the hardening regime), whereas, for the non-associative flow rule adopted in this paper, loss of S-E occurs in the softening as well as in the hardening regime. A comparison with the values of the critical hardening modulus for the loss of ellipticity reported by Rudnicki and Rice (1975) reveals that loss of S-E occurs, for $\xi \neq 0$, well before loss of ellipticity. Finally, it is interesting to note from Fig. 2 that the non-associativity of the flow rule does not necessarily yield, for a given stress state, a critical hardening modulus greater than that corresponding

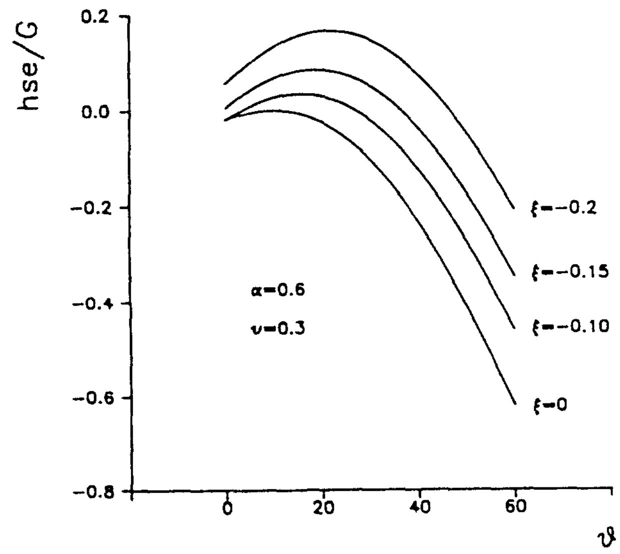

Fig. 3. Critical hardening modulus (normalized in respect to the elastic shear modulus) vs Lode's angle (in degrees).

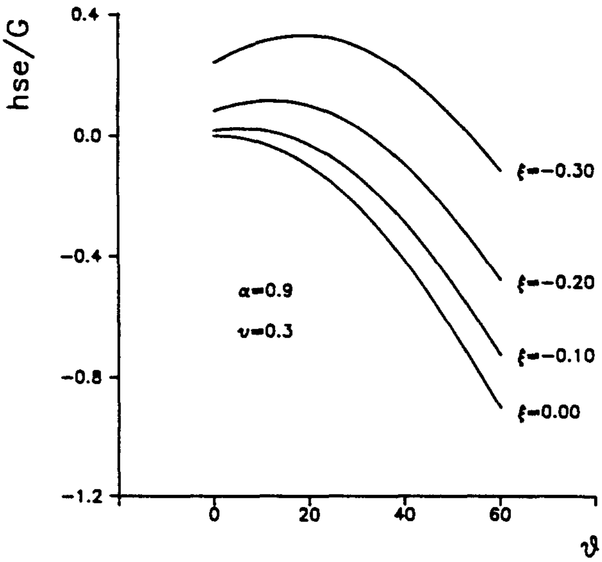

Fig. 4. Critical hardening modulus (normalized in respect to the elastic shear modulus) vs Lode's angle (in degrees).

to the associative flow rule. In fact, in the case of $\alpha = 0.3$ and $\vartheta = 0$, $h_{SE} = -0.093$ for $\xi = 0$ and $h_{SE} = -0.106$ for $\xi = -0.1$.

7. CONCLUSIONS

In this paper, it is shown that in a continuous loading path of an elastoplastic solid with deviatoric normality, strong ellipticity is lost before the snap-back modulus is reached. An analytical solution for the critical plastic modulus for the loss of strong ellipticity is derived for the best chosen comparison solid of Raniecki (1979). This solution is shown to coincide, under the hypothesis of deviatoric normality, with the solution for the loss of strong ellipticity in the comparison solid "in loading".

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APPENDIX A

Solution of the constrained minimization problem (15)

By using the expression (2) of the elastic tensor, the conditions of stationarity of the function (14) are:

$$2\mathbf{A}_F\mathbf{m} - (1-\chi)(2\mu\mathbf{Q}\mathbf{n} + \lambda\mathbf{n}\,\mathrm{tr}\,\mathbf{Q}) - 3\xi\kappa\mathbf{n} = \mathbf{0}, \qquad \varphi = \mathbf{m}\cdot(2\mu\mathbf{Q}\mathbf{n} + \lambda\mathbf{n}\,\mathrm{tr}\,\mathbf{Q}), \tag{A1, A2}$$

and the condition $\mathscr{L}(\mathbf{m}^N, \varphi^N, \chi^N) = 0$, taking into account (A2), is:

$$\mathbf{m}^N\cdot\mathbf{A}_F\mathbf{m}^N - \mathbf{m}^N\cdot(2\mu\mathbf{Q}\mathbf{n} + \lambda\mathbf{n}\,\mathrm{tr}\,\mathbf{Q}) - 3\xi\kappa\mathbf{m}^N\cdot\mathbf{n} = 0. \tag{A3}$$

From (A1), using (2) in the expression (8) of the elastic acoustic tensor, (16) is obtained. By substituting (16) into (A3) the Lagrangean multiplier (18) is obtained. Substitution of (16) into (A2) gives (17).

APPENDIX B

Solution of the problem (35)

Problem (35) is rewritten in the equivalent form:

$$\max_{\mathbf{a}\in\mathscr{D}}\{\varphi^M(\gamma,\mathbf{a}) + \eta(\Sigma_i a_i - 1)\}, \tag{B1}$$

where, from eqn (36)

$$\varphi^M(\gamma,\mathbf{a}) = \frac{G}{2\gamma}\left\{2P_i^2 a_i - \frac{1}{1-v}(P_i a_i)^2 + \frac{2v}{1-v}P_i a_i\,\mathrm{tr}\,\mathbf{P} + \frac{v^2}{(1-v)(1-2v)}\mathrm{tr}^2\,\mathbf{P}\right\}, \tag{B2}$$

in which the indices (summed between 1 and 3) denote components in the principal reference system of **Q** and the components a_i are defined as:

$$a_i = n_i^2. \tag{B3}$$

The set $\mathscr{D}$ in (B1) is therefore defined as:

$$\mathscr{D} = \{\mathbf{a}\in\mathbb{R}^3/(\exists\,\mathbf{n}\in\mathbb{R}^3)(\mathbf{n}\cdot\mathbf{n} = 1 \quad \text{and} \quad a_i = n_i^2, \quad i = 1,2,3)\}. \tag{B4}$$

In Fig. A1, the set $\mathscr{D}$ is represented, in the principal reference system of **Q**, along with the vectors **a** corresponding to the different intervals (49)–(51).

Stationary points inside the set $\mathscr{D}$ correspond to the conditions:

$$P_j P_i a_i - \frac{1-v}{4G}\eta = (1-v)P_j^2 + vP_j\,\mathrm{tr}\,\mathbf{P} \quad (j = 1,2,3), \qquad \Sigma_i a_i = 1. \tag{B5, B6}$$

Solutions of the system (B5)–(B6) are only possible if:

$$P_1 = P_m \quad \text{or} \quad P_1 = P_k \quad \text{or} \quad P_m = P_k. \tag{B7}$$

Conditions (B7) are fully equivalent to

$$Q_1 = Q_m \quad \text{or} \quad Q_1 = Q_k \quad \text{or} \quad Q_m = Q_k. \tag{B8}$$

When one of the equations (B8) holds, e.g. $Q_1 = Q_k \neq Q_m$, the solution of (B5)–(B6) exists if and only if

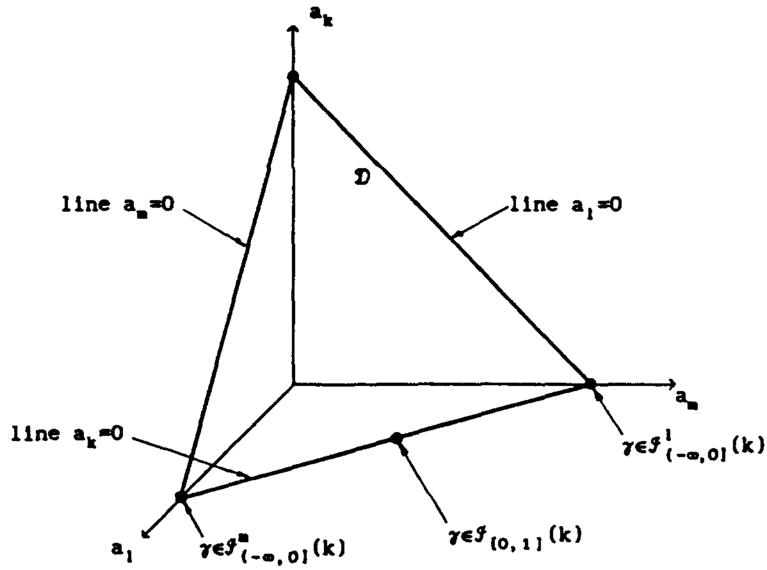

Fig. A1. Representation of the set $\mathcal{D}$ in the principal reference system of Q.

$$0 < -\alpha_l < 1, \tag{B9}$$

where α_l is given by (39). In this case, the function $\varphi^M(\gamma, \mathbf{a})$ is constant on the line $a_l = -\alpha_l$, where it assumes its extremum value and thus, the absolute maximum of $\varphi^M(\gamma, \mathbf{a})$ can be found on a point of the frontier of the set $\mathcal{D}$.

Therefore, the maximum of (35) can in any case be found on the frontier of $\mathcal{D}$. In order to find the maximum of (35) on the frontier of $\mathcal{D}$, let us assume $a_k = 0$. If $a_k = 0$ is substituted into (B2) the stationary conditions on the frontier of $\mathcal{D}$ are:

$$[P_j - \tfrac{1}{2}(P_l + P_m)](P_l a_l + P_m a_m) = v \operatorname{tr} \mathbf{P}[P_j - \tfrac{1}{2}(P_l + P_m)] + (1-v)[P_j^2 - \tfrac{1}{2}(P_l^2 + P_m^2)], \quad (j = l, m), \qquad a_l + a_m = 1. \tag{B10, B11}$$

The solutions of (B10)–(B11) are:

$$a_l = 0,\ a_m = 1 \quad \text{or} \quad a_m = 0,\ a_l = 1, \quad \text{if} \quad P_l = P_m, \tag{B12}$$

$$\begin{cases} a_l = -\alpha_l,\ a_m = -\alpha_m & \text{if} \quad \alpha_l, \alpha_m \in [-1, 0] \\ a_l = 0,\ a_m = 1 & \text{if} \quad \alpha_l \geqslant 0 \\ a_l = 1,\ a_m = 0 & \text{if} \quad \alpha_l \leqslant -1 \end{cases}, \quad \text{if} \quad P_l \neq P_m. \tag{B13}$$

Solutions (B12)–(B13), using (B3), yield (38). Expression (38), substituted into (36) gives:

$$\varphi^M(\gamma, k) = -\frac{G}{2\gamma}(1+v)P_k^2 - \frac{G}{2\gamma(1-v)}\left[\frac{\langle \alpha_m \rangle}{\alpha_m}(P_m + vP_k)^2 + \frac{\langle \alpha_l \rangle}{\alpha_l}(P_l + vP_k)^2\right] + \frac{G}{2\gamma}\left[\frac{v}{1-2v}\operatorname{tr}^2 \mathbf{P} + \operatorname{tr} \mathbf{P}^2\right]. \tag{B14}$$

Equation (42) is obtained by substituting the definition (37) of **P** into expression (B14).

APPENDIX C

For two different values p and q of k, function (42) (see also B14) assumes the same values in the interval:

$$\mathcal{I}^{pq}_{(-\infty,0]} = \mathcal{I}^{p}_{(-\infty,0]}(q) \cap \mathcal{I}^{q}_{(-\infty,0]}(p). \tag{C1}$$

Moreover, the interval (C1) is empty or contains more than one point. Note that the two functions $\varphi^M(\gamma, p)$ and $\varphi^M(\gamma, q)$ have the same derivative (in all points and) at the extreme points of interval (C1), since they are coincident in the interval (C1).

Proof. If $\gamma \in \mathcal{I}^{pq}_{(-\infty,0]}$, from (C1) and (53)–(54) follows $\alpha_p(k = q) \geqslant 0$ and $\alpha_q(k = p) \geqslant 0$. It is therefore easily verified, using (B14), that

$$\varphi^{\mathrm{M}}(\gamma, p) - \varphi^{\mathrm{M}}(\gamma, q) = 0 \tag{C2}$$

and thus the function (42) assumes for $k = p$ and $k = q$ the same values in the interval (C1). Moreover, if the interval (C1) reduces to a single-point, one of the following conditions proves to be verified:

$$Q_t = Q_q \quad \text{or} \quad Q_t = Q_p \quad \text{or} \quad Q_q = Q_p = -\xi/(1+\gamma), \tag{C3}$$

where (t, p, q) denotes a permutation of $(1, 2, 3)$. Conditions (C3) imply

$$\mathscr{I}^q_{(-\infty,0]}(p) = \mathbb{R}^+ \quad \text{or} \quad \mathscr{I}^p_{(-\infty,0]}(q) = \mathbb{R}^+, \tag{C4}$$

thus the interval (C1) coincides with one of the two intervals $\mathscr{I}^p_{(-\infty,0]}(q)$, $\mathscr{I}^q_{(-\infty,0]}(p)$ that are empty or contain more than one point. Therefore (C1) does not reduce to one single point.

APPENDIX D

If $\bar{\gamma} \notin \mathscr{I}^p_{(-\infty,0]}(q) \cup \mathscr{I}^q_{(-\infty,0]}(p)$ and $\bar{\gamma} \in \mathscr{I}_{[0,1]}(p) \cup \mathscr{I}_{[0,1]}(q)$ then $\varphi^{\mathrm{M}}(\bar{\gamma}, p) = \varphi^{\mathrm{M}}(\bar{\gamma}, q)$ is possible only if an extremal point of $\varphi^{\mathrm{M}}(\bar{\gamma}, \mathbf{a})$ [see (B2)] is inside the set $\mathscr{D}$ defined by (B4). In Appendix **B** it has been shown that this circumstance may be verified only when tensor **Q** has at least one multiple eigenvalue [condition (B8)]. If $\bar{\gamma} \notin \mathscr{I}^{pq}_{(-\infty,0]}(q)$ and $\bar{\gamma} \in \mathscr{I}^p_{(-\infty,0]}(q) \cap \mathscr{I}^t_{(-\infty,0]}(p)$ or $\bar{\gamma} \in \mathscr{I}^q_{(-\infty,0]}(p) \cap \mathscr{I}^t_{(-\infty,0]}(q)$ or $\bar{\gamma} \in \mathscr{I}^p_{(-\infty,0]}(q) \cap \mathscr{I}_{[0,1]}(p)$ or $\bar{\gamma} \in \mathscr{I}^q_{(-\infty,0]}(p) \cap \mathscr{I}_{[0,1]}(p)$, points of intersection are not possible. If $\bar{\gamma} \in \mathscr{I}^t_{(-\infty,0]}(q) \cap \mathscr{I}^t_{(-\infty,0]}(p)$ the equality

$$\varphi^{\mathrm{M}}(\bar{\gamma}, p) - \varphi^{\mathrm{M}}(\bar{\gamma}, q) = 0, \tag{D1}$$

becomes:

$$(P_p - P_q)(P_p + P_q + 2\nu P_t) = 0. \tag{D2}$$

Condition (D2) implies that tensor **Q** either has one multiple eigenvalue or:

$$P_p + P_q + 2\nu P_t = 0. \tag{D3}$$

Taking into account (D3) and (B14) [see also (42)], the following inequality is readily obtained:

$$\varphi^{\mathrm{M}}(\bar{\gamma}, t) > \varphi^{\mathrm{M}}(\bar{\gamma}, p) = \varphi^{\mathrm{M}}(\bar{\gamma}, q). \tag{D4}$$